Bounds for Different Spreads of Line and Total Graphs

Enide Andrade

CIDMA-Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro, 3810-193, Aveiro, Portugal.

Eber Lenes

Grupo de Investigación Deartica, Universidad del Sinú. Elías Bechara Zainúm, Cartagena, Colombia.

Exequiel Mallea-Zepeda

Departamento de Matemática, Universidad de Tarapacá, Arica, Chile.

*María Robbiano**

Departamento de Matemáticas, Universidad Católica del Norte, Av. Angamos 0610 Antofagasta, Chile.

Jonnathan Rodríguez Z.

Departamento de Matemáticas, Universidad de Antofagasta,Av.Angamos 601, Antofagasta, Chile

---

Abstract

In this paper we explore some results concerning the spread of the line and the total graph of a given graph. A sufficient condition for the spread of a unicyclic graph with an odd girth to be at most the spread of its line graph is presented. Additionally, we derive an upper bound for the spread of the line graph of graphs on n vertices having a vertex (edge) connectivity at most a positive integer k. Combining techniques of interlacing of eigenvalues, we derive lower bounds for the Laplacian and signless Laplacian spread of the

*Corresponding author

Email addresses: `enide@ua.pt` (Enide Andrade), `elenes@unisinucartagena.edu.co` (Eber Lenes), `emallea@uta.cl` (Exequiel Mallea-Zepeda), `mrobbiano@ucn.cl` (María Robbiano), `jrodriguez01@ucn.cl` (Jonnathan Rodríguez Z.)

total graph of a connected graph. Moreover, for a regular graph, an upper and lower bound for the spread of its total graph is given.

Keywords: Matrix Spread; Graph Spread; Q-spread; Total Graph; Line Graph; Connectivity
2000 MSC: 05C50, 15A18

---

1. Introduction

Let G be a simple undirected graph with vertex set $V(G)$ of cardinality n and edge set $E(G)$ of cardinality m. We say that G is an (n,m) graph. After the labeling of vertices, a vertex is named by its label and an edge with end vertices i and j is denoted by ij. The degree of i is denoted by $d_G(i)$ (or simply $d(i)$). A graph G is called r-regular if each vertex has degree r. For a finite set U, $|U|$ denotes its cardinality. If $U \subseteq V(G)$, $G-U$ denotes the subgraph of G induced by $V(G)\backslash U$. The diameter of a graph G is the greatest distance between any two vertices of G and it will be denoted by $diam(T)$. For two disjoint graphs G_1 and G_2, the join of G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from their union including all edges between the vertices in G_1 and the vertices in G_2. The adjacency matrix, Laplacian matrix and signless Laplacian matrix associated with a graph G are denoted by $A(G)$, $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively, where $D(G)$ is the diagonal matrix of vertex degrees. The eigenvalues of a graph G are the eigenvalues of its adjacency matrix, denoted by $\lambda_i = \lambda_i(G)$ and ordered in nonincreasing way as $\lambda_1 \geq \ldots \geq \lambda_n$. Moreover, the eigenvalues of $L(G)$ and $Q(G)$ are also ordered in nonincreasing way as follows $\mu_1 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$, and $q_1 \geq \ldots \geq q_n$, respectively. These are called *Laplacian eigenvalues* and *signless Laplacian eigenvalues* of G, respectively. In this work, K_n and C_n denote the complete graph and the cycle of order n, respectively. Moreover, I_m denotes the identity matrix of order m and, for a matrix A, A^t denotes its transpose. The line graph $\mathcal{L}(G)$ is the graph whose vertex set are the edges in G, where two vertices are adjacent if the corresponding edges in G have a common vertex. Here, if G is an (n,m) graph $\lambda_1(\mathcal{L}) \geq \lambda_2(\mathcal{L}) \geq \ldots \geq \lambda_m(\mathcal{L})$ will denote the eigenvalues of $\mathcal{L} = \mathcal{L}(G)$. The total graph $\mathcal{T}(G)$ is the graph whose vertex set corresponds to the vertices of $G \cup \mathcal{L}(G)$ and two vertices are adjacent in $\mathcal{T}(G)$ if their corresponding elements are either adjacent in $G \cup \mathcal{L}(G)$ or incident in G. In [3] the author presented a characterization of the structure of regular total graphs as well as other properties. In [2] non-regular graphs were considered and it was presented a method that enables to determine whether a graph is total or not. Moreover, a relationship between the spectra of a regular graph and its total graph was presented by Cvetković in [9]. Many graph theoretical parameters have been used to describe the stability of communication networks. Tenacity, [8], is one of these parameters, which shows not only the difficulty to break down the network but also the damage that has been caused. For a complete graph K_n its tenacity is defined as n, and for a non-complete connected graph G, its tenacity is defined as $\min\{\frac{|S|+\gamma(G-S)}{\omega(G-S)}\}$ where the minimum is taken over all the cutsets $\mathcal{S}$ of $V(G)$, $\omega(G-S)$ is the number of components of $G-S$ and $\gamma(G-S)$ is the number of vertices in the largest component of the graph induced by $G-S$. Total graphs are the largest graphs formed by the adjacent relations of elements of a given graph. Thus, total graphs are highly recommended for the design of interconnection networks. For instance, in [24] the authors determine the tenacity of the total graph of a path, cycle and complete bipartite graph, and thus give a lower bound of the tenacity for the total graph of an arbitrary graph. It is also worth to recall that total graphs are generalizations of line graphs.

The paper is organized as follows. In Section 2 the definitions of spread, Laplacian spread, signless Laplacian spread are presented. A simple upper bound for the spread of the line graph as function of the Zagreb index and the number of edges of the graph is obtained. In Section 3, relations among the spread of the line graph and the signless Laplacian spread are given. A sufficient condition for the spread of a unicyclic graph with an odd girth to be at most the spread of its line graph is given. Moreover, it is derived an upper bound for the spread of the line graph of graphs with n vertices having a vertex (edge) connectivity less than or equal to k. This bound is attained if and only if $G \cong K_1 \vee (K_k \cup K_{n-k-1})$, where K_k is the complete graph of order k. In Section 4, using the Laplacian and signless Laplacian matrices of the total graph of a connected graph and applying the interlacing of eigenvalues, due to Haemers [17], we obtain lower bounds for the spread, signless Laplacian spread and Laplacian spread of total graphs. Moreover, in the case of a regular graph G we present an upper and lower bound for the spread of the total graph associated to G.

2. Preliminaries

In this section, we list some of the definitions of different spreads and previously known results that will be needed throughout the paper.

It is known that the matrices $Q(G)$ and $2I_m + A(\mathcal{L}(G))$ share the same nonzero eigenvalues, see for instance [11]. As a consequence, we have the following result.

Lemma 1. [4] Let G be an (n, m) graph with $m \geq 1$ edges. Let q_i be the *i-th* greatest signless Laplacian eigenvalue of G and $\lambda_i(\mathcal{L}(G))$ the *i-th* greatest eigenvalue of its line graph $\mathcal{L}(G)$. Then

$$q_i = \lambda_i(\mathcal{L}(G)) + 2,$$

for $i = 1, 2, \ldots, k$, where $k = \min\{n, m\}$. In addition, if $m > n$, then $\lambda_i(\mathcal{L}(G)) = -2$ for $i \geq n+1$ and if $n > m$, then $q_i = 0$ for $i \geq m+1$.

The Zagreb index of G, $Z_g(G)$, (see [16]), is defined as

$$Z_g(G) = \sum_{i \in V(G)} d^2(i).$$

If G is a graph with m edges, it is a simple exercise to verify that the number of edges of $\mathcal{L}(G)$ is given by $\sum_{i \in V(G)} \frac{d(i)(d(i)-1)}{2}$ and then, the number of edges of $\mathcal{L}(G)$ can be given by

$$\theta = \frac{Z_g(G)}{2} - m. \tag{1}$$

The *spread* of an $n \times n$ complex Hermitian matrix M with eigenvalues $\lambda_1, \ldots, \lambda_n$ is

$$S(M) = \max_{i,j} |\lambda_i - \lambda_j|,$$

where the maximum is taken over all pairs of distinct eigenvalues of M.

The *spread of the graph* G, [14], with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ is defined as the spread of its adjacency matrix, that is:

$$S(G) = \lambda_1 - \lambda_n.$$

Let $K_{a,b}$ be the complete bipartite graph where the bipartition of its vertex set has a vertices in one subset and b vertices in the other. The next result can be seen in [14].

Lemma 2. [14] Let G be an (n,m) graph. Then

$$S(G) \leq \lambda_1 + \sqrt{2m - \lambda_1^2} \leq 2\sqrt{m}.$$

Equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $m = 0$ or $G = K_{a,b}$, for some a, b with $m = ab$ and $a + b \leq n$.

The *signless Laplacian spread* or *Q-spread* of G is defined in [21] by

$$S_Q(G) = S(Q(G)) = q_1 - q_n, \tag{2}$$

where $q_1 \geq q_2 \geq \ldots \geq q_n$ are the signless Laplacian eigenvalues of G.

Some results on the Q-spread of a graph can be found for instance in Liu and Liu [21] and Oliveira *et al.* [22].

As μ_n is always zero, the *Laplacian spread* of G is defined in a slightly different way, see e.g., [12],

$$S_L(G) = \mu_1 - \mu_{n-1}, \tag{3}$$

where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$ are the Laplacian eigenvalues of G.

Attending to the relation between the signless Laplacian eigenvalues and the eigenvalues of $\mathcal{L}$ presented in Lemma 1, the expression for its number of edges given by equation (1), and Lemma 2 the following upper bound is a simple consequence:

$$S_{\mathcal{L}}(G) \leq q_1 - 2 + \sqrt{Z_g(G) - 2m - (q_1 - 2)^2} \leq 2\sqrt{\frac{Z_g(G)}{2} - m}. \tag{4}$$

In the next section it will be proved that $S(G) \leq S_{\mathcal{L}}(G)$ for some (n,m) connected graphs.

3. Bounds for the spread of the line graph

In this section we present a sufficient condition for the spread of a unicyclic graph with an odd girth to be at most the spread of its line graph. The condition depends on the girth and the maximum diameter among the induced subtrees of the graph. Additionally, considering the family of connected graphs whose vertex (edge) connectivity is at most k, with $k > 0$, it is given a tight upper bound for the spread of the line graph of a graph in

this family. In this case, it it is shown the class of graphs that attains the referred upper bound.
Next Lemma gives relations between $S_Q(G)$ and $S_{\mathcal{L}}(G)$. It is a direct consequence of Lemma 1.

Lemma 3. *Let G be an (n,m) graph.*

1. *If $m=n$, then $S_{\mathcal{L}}(G)=q_1-q_n=S_Q(G)$.*
2. *If $m>n$, then $S_{\mathcal{L}}(G)=q_1\geq S_Q(G)$, with equality if and only if G has a bipartite component.*
3. *If $m<n$, then $S_{\mathcal{L}}(G)=q_1-q_m\leq q_1=S_Q(G)$, with equality, for example, for $G=C_n\cup K_2$, when n is even.*

Let G be an arbitrary graph. In [1] it was established that $S(G)\leq q_1$, with equality if and only if G is a bipartite regular graph. If G is a unicyclic connected graph with even cycle, from Lemma 3, $S_{\mathcal{L}}(G)=S_Q(G)=q_1$. Thus $S(G)\leq S_{\mathcal{L}}(G)=S_Q(G)$.

When G is a unicyclic graph with an odd cycle the partial result at Theorem 6 is proved. Before proceeding to the proof of Theorem 6 we need some previous results. Let T be a tree. The second smallest Laplacian eigenvalue of T is referred as the *algebraic connectivity of T*, it is denoted by $a(T)$. Grone *et al.*, in [15] proved that

$$a\left(T\right)\leq 0.49$$

holds for any tree T with at least six vertices. Moreover, the same authors obtained an upper bound for the algebraic connectivity of a tree in function of $diam(T)$,

$$a\left(T\right)\leq 1-\cos\frac{\pi}{diam(T)+1}. \tag{5}$$

On the other hand, the following result is in [5].
Recall that the *girth* of a graph G is the length of a shortest cycle in G.

Theorem 4. *[5] Let e be an edge of the graph G. Let $q_1,\ldots,q_n$ and $s_1,\ldots,s_n$ be the signless Laplacian eigenvalues of G and of $G-e$, respectively. Then*

$$0\leq s_n\leq q_n\leq s_{n-1}\leq\ldots\leq s_2\leq q_2\leq s_1\leq q_1. \tag{6}$$

Lemma 5. *[7, 10] Let G be a graph with n vertices. Then*

$$2\lambda_1 \leq q_1, \tag{7}$$

where λ_1 is the spectral radius of G. The equality holds if and only if G is regular.

Theorem 6. *Let G be a connected unicyclic graph with odd girth g and whose maximum diameter among the induced trees of G is h. Let $\lambda_1(G)$ and $\lambda_n(G)$ be the largest and smallest eigenvalue of G, respectively. If*

$$\lambda_n(G) \geq 1 - \cos\frac{\pi}{D_0 + 1} - \lambda_1(G), \tag{8}$$

where

$$D_0 = \frac{g+1}{2} + h,$$

then

$$S(G) \leq S_{\mathcal{L}}(G). \tag{9}$$

Proof. Note that the girth of G corresponds, in this case, to the number of vertices of the induced cycle of G. If G is a cycle then G is regular. Then, $S(G) = S_Q(G)$. From Lemma 3(1) the relation in (9) holds in the equality. If G is not a cycle let us consider T as an induced tree of G such that $diam(T) = h$. Consider v as the vertex within the cycle of G which is the root vertex of T. Let w be a vertex within the cycle of G placed at the end of the path of length $\frac{g+1}{2}$ starting in v. Let u be the neighbor of w placed at the end of the path of length $1 + \frac{g+1}{2}$ starting in v within the cycle of G. Let e be the edge wu. Then the deletion of e yields a tree T_0 with root vertex w and diameter $D_0 = \frac{g+1}{2} + h$ (see Figure 1). Let $q_1, \ldots, q_n$ and $s_1, \ldots, s_n$ be the signless Laplacian eigenvalues of G and of $G - e$, respectively. By (6) one can see that

$$q_n \leq s_{n-1} = a(T_0) \leq 1 - \cos\frac{\pi}{D_0 + 1}. \tag{10}$$

From (7), (8) and (10) we have

$$q_1 - q_n \geq q_1 - 1 + \cos\frac{\pi}{D_0 + 1} \geq 2\lambda_1 - 1 + \cos\frac{\pi}{D_0 + 1} \geq \lambda_1 - \lambda_n.$$

But this means that

$$S(G) \leq S_Q(G).$$

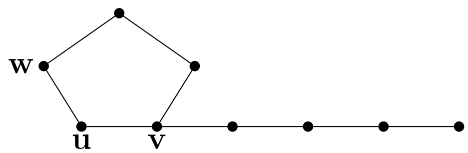

Figure 1: $g = 5,\ h = 4,\ D_0 = diam\,(T_0) = \frac{g+1}{2} + h = 3 + 4 = 7.$

Hence, by Lemma 3(1) again, the relation in (9) holds. ∎

The next example illustrates the previous theorem.

Example 7. *For the graph G depicted in the Figure 1, we have*

$$\lambda_1(G) = 2.17,\ \ \lambda_n(G) = -2, D_0 = 7,\ \cos\frac{\pi}{D_0+1} = 0.9239.$$

Then, the condition in (8) becomes

$$-2 \geq 1 - 0.9239 - 2.17 = -2.0939.$$

Moreover, $S_{\mathcal{L}}(G) = S_Q(G) = 4.47$ and $S(G) = 4.17$, which check our result.

In Theorem 6 we consider an induced tree, that is to say a tree whose root vertex is a vertex of the cycle. Since an edge of the cycle was removed to obtain the condition in (8) a spanned subtree can also be considered. The Remark below refers to this case.

Remark 8. *The result of Theorem 6 can be obtained (and in some cases the condition in (8) can be weakened) by consider D_0 as the maximum of the diameters of the spanned trees of the unicycle graph G.*

In what follows it is characterized the graph with maximum spread of its line graph, into the family of connected graphs with vertex (edge) connectivity at most k, where k is a given positive integer. Some preliminary results are previously presented.

In [6, Theorem 5], the spectrum of the adjacency matrix of the H-*join* of regular graphs is obtained. Consider the graph $K_i \vee (K_k \cup K_{n-k-i})$. As this graph can be seen as the P_3-join of the family of graphs $\{K_i, K_k, K_{n-k-i}\}$, from [6, Theorem 5], Corollary 9 below is immediate.

Corollary 9. *The eigenvalues of the line graph of $K_i \vee (K_k \cup K_{n-k-i})$ are*

$$n + \frac{k}{2} - 4 + \frac{1}{2}\sqrt{(2n-k)^2 + 16i(k-n+i)},$$
$$n + \frac{k}{2} - 4 - \frac{1}{2}\sqrt{(2n-k)^2 + 16i(k-n+i)},$$

both with multiplicity 1 *and*

$$n-4, k+i-4, n-i-4, -2,$$

with multiplicities $k, i-1, n-k-i-1$ and $m-n$, respectively.

Theorem 10. *[20] For $n \geq 6$ and $k \geq 3$ the line graph of $K_i \vee (K_k \cup K_{n-k-i})$ has exactly $m-n$ negative eigenvalues, all equal to -2.*

The following result is a direct consequence of Corollary 9, Theorem 10 and the definition of spread of a graph.

Proposition 11. *Let $n \geq 6$ and $k \geq 3$. Then*

$$S_{\mathcal{L}}\left(K_i \vee \left(K_k \cup K_{n-k-i}\right)\right) = n - 2 + \frac{1}{2}k + \sqrt{\left(2n-k\right)^2 + 16i\left(k-n+i\right)}.$$

It is known that the spectral radius of a nonnegative irreducible matrix increases if any of its entries increases. From this fact, we have the following result.

Lemma 12. *Let G be a connected graph then*

$$q_1(G) < q_1(G+e),$$

where $G+e$ denotes the graph that results from G adding an edge e.

From Lemma 12 above the following result is immediate.

Lemma 13. *Let G be a connected graph. Then*

$$\lambda_1(\mathcal{L}(G)) < \lambda_1(\mathcal{L}(G+e)).$$

The following result is a direct consequence of Cauchy's Interlacing Theorem [19], and Lemma 13.

Theorem 14. *Let G be a connected graph. Then*

$$S_{\mathcal{L}}(G) < S_{\mathcal{L}}(G+e).$$

The *vertex connectivity* (or just *connectivity*) of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices of G whose deletion disconnects G. Let $\mathcal{F}_n$ be the family of connected graphs on n vertices. Let

$$\mathcal{V}_n^k = \{G \in \mathcal{F}_n : \kappa(G) \leq k\}.$$

The following result characterizes the graph with maximum spread of its line graph into the family of connected graphs with vertex connectivity at most k, where k is a given positive integer.

Theorem 15. *Let $n \geq 6$ and $k \geq 3$. Let $G \in \mathcal{V}_n^k$. Then,*

$$S_{\mathcal{L}}(G) \leq n - 2 + \frac{1}{2}k + \sqrt{(2n-k)^2 + 16(k-n+1)} \tag{11}$$

with equality if and only if $G \cong K_1 \vee (K_k \cup K_{n-k-1})$.

Proof. Let $G \in \mathcal{V}_n^k$ be such that $\mathcal{L}(G)$ has the largest spread among all the graphs $\mathcal{L}(H)$ with $H \in \mathcal{V}_n^k$. Let $U \subset V(G)$, such that $|U| \leq k$ and $G - U$ is a disconnected graph. Let $X_1, X_2, \ldots, X_l$, be the connected components of $G - U$. We claim that $l = 2$. If $l > 2$ then we can construct a graph $H = G + e$ where e is an edge connecting a vertex in X_1 with a vertex in X_2. Clearly, $H \in \mathcal{V}_n^k$. By Theorem 14, $S_{\mathcal{L}}(G) < S_{\mathcal{L}}(H)$, which is a contradiction. Therefore $l = 2$, that is, $G - U = X_1 \cup X_2$. Recall that $|U| \leq k$. Now, we claim that $|U| = k$. Suppose $|U| < k$, then we construct a graph $H = G + e$ where e is an edge joining a vertex $u \in V(X_1)$ with a vertex $v \in V(X_2)$. We see that $H - U$ is a connected graph and the deletion of the vertex u disconnects $H - U$. This tells us that $H \in \mathcal{V}_n^k$. By Theorem 14, $S_{\mathcal{L}}(G) < S_{\mathcal{L}}(H)$, which is a contradiction. Then, $|U| = k$. Therefore, $G - U = X_1 \cup X_2$ and $|U| = k$. Let $|X_1| = i$ then $|X_2| = n - k - i$.

We claim that repeated applications of Theorem 14 enable us to write

$$G(i) \cong K_i \vee (K_k \cup K_{n-k-i}) \cong G$$

for some $1 \leq i \leq \lfloor \frac{n-k}{2} \rfloor$. In fact, this means that if Y_3 is the induced subgraph of G obtained from the vertices in U then, there would be an edge

$$e \in [E(\overline{Y_1} \vee \overline{Y_3}) \cup E(\overline{Y_2} \vee \overline{Y_3})] - E(G).$$

Therefore, it is possible to construct a new graph $H = G + e$. Clearly, $H \in \mathcal{V}_n^k$. By Theorem 12, $S_{\mathcal{L}}(G) < S_{\mathcal{L}}(H)$ which is a contradiction.

Until this point, we have proved $S_{\mathcal{L}}(G) \leq S_{\mathcal{L}}(G(i))$, for all $G \in \mathcal{V}_n^k$. We now search for a value of i for which $S_{\mathcal{L}}(G(i))$ is maximum.

From Theorem 11, it is obtained

$$S_{\mathcal{L}}(G(i)) = n - 2 + \frac{1}{2}k + \sqrt{(2n-k)^2 + 16i(k-n+i)}.$$

Define the function,

$$g(x) = n - 2 + \frac{1}{2}k + \sqrt{(2n-k)^2 + 16x(k-n+x)}$$

where $1 \leq x \leq \lfloor \frac{n-k}{2} \rfloor$. In this interval g is a strictly decreasing function. Consequently $S_{\mathcal{L}}(G) \leq S_{\mathcal{L}}(G(1))$ for all $G \in \mathcal{V}_n^k$. Moreover, since $G(1) \cong K_k \vee (K_1 \cup K_{n-k-1})$ and $S_{\mathcal{L}}(G(1)) = n-2+\frac{1}{2}k+\sqrt{(2n-k)^2+16(k-n+1)}$ the equality in (11) holds if and only if $G \cong K_1 \vee (K_k \cup K_{n-k-1})$. □

We recall now the definition of *edge-connectivity* of G, denoted here by $\varepsilon(G)$, as the minimum number of edges whose deletion disconnects G. Note also that in graphs that represent communication or transportation networks, the edge-connectivity is important to measure the network reliability.

Let

$$\mathcal{E}_n^k = \{G \in \mathcal{F}_n : \varepsilon(G) \leq k\}.$$

It is well known that $\kappa(G) \leq \varepsilon(G) \leq \delta(G)$, where $\delta(G)$ denotes the minimum degree of G, see [18, 23].

Let $\Delta_n^k = \{G \in \mathcal{F}_n : \delta(G) \leq k\}$. Then, $\Delta_n^k \subseteq \mathcal{V}_n^k$. Moreover, the graph $K_1 \vee (K_k \cup K_{n-k-1})$ have minimum degree k. Then, attending to Theorem 15, we can also obtain the following result.

Corollary 16. *Let $n \geq 6$ and $k \geq 3$. Let $G \in \Delta_n^k$. Then,*

$$S_{\mathcal{L}}(G) \leq n - 2 + \frac{1}{2}k + \sqrt{(2n-k)^2 + 16(k-n+1)}$$

with equality if and only if $G \cong K_1 \vee (K_k \cup K_{n-k-1})$.

The same result is obtained if we replace Δ_n^k by $\mathcal{E}_n^k$ and the equality is attained in the same graph.

4. Lower bounds for different spreads of total graphs

In this section we present some lower bounds for different spreads of total graphs. The tools used here were the interlacing of the eigenvalues [17, p. 594] and the definition of equitable partitions that we recall below.

For $1 \leq i, j \leq k$, let us consider the $n_i \times n_j$ matrices M_{ij}. Let $n = \sum_{i=1}^{k} n_i$ and suppose that $M_{ij} = M_{ji}^t$, for all (i, j). We consider the partitioning of the $n \times n$ symmetric matrix into blocks

$$M = \left(M_{ij}\right)_{1 \leq i,j \leq k}. \tag{12}$$

Let us denote by $\mathbb{J}_{pq}$ the all ones matrix of order $p \times q$ and simply by $\mathbb{J}_p$ the all ones vector of order $p \times 1$. The quotient matrix $\overline{M} = (m_{ij})$ of M is the $k \times k$ matrix whose (i, j)-entry is the average of the row sums of M_{ij}. More precisely

$$m_{ij} = \frac{1}{n_i}\left(\mathbb{J}_{n_i}^t M_{ij} \mathbb{J}_{n_j}\right), \qquad \text{for } 1 \leq i, j \leq k. \tag{13}$$

The partitioning into blocks of M is called regular (or equitable) if each block M_{ij} of M has constant row sum. Note that in this case $\overline{M}$ corresponds to the row sums matrix. According to [17, Corollary 2.3], if M is regular, then all the eigenvalues of $\overline{M}$ are eigenvalues of M.

Let G be an (n, m) graph. We recall that the incidence matrix of a graph H is a matrix R whose rows and columns are indexed by the vertices and edges of H, respectively. The (i, j)- entry of R is $r_{ij} = 0$ if i is not incident with j and $r_{ij} = 1$ if j is incident with i.

The adjacency matrix of the total graph $\mathcal{T}(G)$ is given by

$$A\left(\mathcal{T}\left(G\right)\right) = \begin{bmatrix} A\left(G\right) & R \\ R^t & A\left(\mathcal{L}\left(G\right)\right) \end{bmatrix}$$

where R is the incidence matrix of G.

Directly from the definition, we conclude that for $u \in V\left(\mathcal{T}\left(G\right)\right)$

$$d_{\mathcal{T}(G)}\left(u\right) = \begin{cases} 2d_G\left(u\right) & \text{if} \quad u \in V\left(G\right) \\ d_{\mathcal{L}(G)}\left(u\right) + 2 & \text{if} \quad u \in V\left(\mathcal{L}(G)\right) \end{cases}.$$

Hence, the diagonal matrix of vertex degrees of $\mathcal{T}\left(G\right)$ is

$$D\left(\mathcal{T}\left(G\right)\right) = \begin{bmatrix} 2D\left(G\right) & 0 \\ 0 & D\left(\mathcal{L}(G)\right) + 2I_m \end{bmatrix}.$$

Remark 17. *Let $\delta(G)$ and $\Delta(G)$ the minimum and maximum vertex degree of a graph G, respectively. It is immediate that*

$$\delta(\mathcal{T}(G)) = 2\delta(G) \tag{14}$$

$$\Delta(\mathcal{T}(G)) = 2\Delta(G). \tag{15}$$

Now a lower bound for the signless Laplacian spread of total graphs is presented.

Theorem 18. *Let G be an (n,m) connected graph. Then,*

$$S_Q\left(\mathcal{T}\left(G\right)\right) \geq 2\sqrt{\left(\frac{3m}{n} - \frac{Z_g(G)}{m}\right)^2 + \frac{10m}{n} - \frac{2Z_g(G)}{m} + 1}.$$

Proof. According to [17, Corollary 2.3], considering the signless Laplacian matrix $Q\left(\mathcal{T}\left(G\right)\right)$

$$\begin{aligned} Q\left(\mathcal{T}\left(G\right)\right) &= D\left(\mathcal{T}\left(G\right)\right) + A\left(\mathcal{T}\left(G\right)\right) \\ &= \begin{bmatrix} 2D\left(G\right) & 0 \\ 0 & D\left(\mathcal{L}(G)\right) + 2I_m \end{bmatrix} + \begin{bmatrix} A\left(G\right) & R \\ R^t & A\left(\mathcal{L}\left(G\right)\right) \end{bmatrix} \\ &= \begin{bmatrix} Q\left(G\right) & 0 \\ 0 & Q\left(\mathcal{L}(G)\right) \end{bmatrix} + \begin{bmatrix} D\left(G\right) & R \\ R^t & 2I_m \end{bmatrix}, \end{aligned} \tag{16}$$

the quotient matrix of $Q\left(\mathcal{T}\left(G\right)\right)$ becomes

$$\overline{M}_Q := \overline{M}\left(Q\left(\mathcal{T}\left(G\right)\right)\right) = \begin{bmatrix} \frac{6m}{n} & \frac{2m}{n} \\ 2 & 2 + \frac{4\theta}{m} \end{bmatrix},$$

where θ stands for the number of edges of the line graph. The characteristic equation of $\overline{M}_Q$ is

$$\lambda^2 - 2\lambda\left(1 + \frac{3m}{n} + \frac{2\theta}{m}\right) + \frac{8}{n}\left(m + 3\theta\right) = 0.$$

Solving this equation, we have

$$\begin{aligned} \lambda_1\left(\overline{M}_Q\right) &= \left(\frac{3m}{n} + \frac{2\theta}{m} + 1\right) + \sqrt{\left(\frac{3m}{n}\right)^2 + \left(\frac{2\theta}{m}\right)^2 + 1 - \frac{12\theta}{n} - \frac{2m}{n} + \frac{4\theta}{m}}, \\ \lambda_2\left(\overline{M}_Q\right) &= \left(\frac{3m}{n} + \frac{2\theta}{m} + 1\right) - \sqrt{\left(\frac{3m}{n}\right)^2 + \left(\frac{2\theta}{m}\right)^2 + 1 - \frac{12\theta}{n} - \frac{2m}{n} + \frac{4\theta}{m}}. \end{aligned}$$

Therefore,

$$S_Q(\mathcal{T}(G)) \geq 2\sqrt{\left(\frac{3m}{n}\right)^2 + \left(\frac{2\theta}{m}\right)^2 + 1 - \frac{12\theta}{n} - \frac{2m}{n} + \frac{4\theta}{m}}.$$

Recalling from (1) the number of edges of the line graph of a graph G with n vertices and m edges, the result follows. $\square$

Moreover, the following lower bound for the spread of total graphs is obtained.

Theorem 19. *Let G be a connected graph on n vertices and m edges. Then*

$$S(\mathcal{T}(G)) \geq \sqrt{\left(\frac{2m^2 + n\left(Z_g(G) - 2m\right)}{mn}\right)^2 - \frac{8\left(Z_g(G) - 4m\right)}{n}}.$$

Proof. The quotient matrix of $A(\mathcal{T}(G))$ is given by

$$\overline{M}_A := \overline{M}\left(A\left(\mathcal{T}(G)\right)\right) = \begin{bmatrix} \frac{2m}{n} & \frac{2m}{n} \\ 2 & \frac{Z_g(G)-2m}{m} \end{bmatrix}.$$

Then, the characteristic equation of $\overline{M}_A$ is

$$\left(\lambda - \frac{2m}{n}\right)\left(\lambda - \frac{Z_g(G) - 2m}{m}\right) - \frac{4m}{n} = 0.$$

Solving the equation, we have

$$\lambda_{\pm}\left(\overline{M}_A\right) = \frac{1}{2}\psi \pm \sqrt{\psi^2 - \frac{8(Z_g(G) - 4m)}{n}},$$

where $\psi := \frac{2m^2+n((Z_g(G)-2m)}{mn}$.

Therefore,

$$S(\mathcal{T}(G)) \geq \sqrt{\left(\frac{2m^2 + n\left(Z_g(G) - 2m\right)}{mn}\right)^2 - \frac{8\left(Z_g(G) - 4m\right)}{n}}.$$

$\square$

The Laplacian matrix of $\mathcal{T}(G)$ is

$$\begin{aligned} L(\mathcal{T}(G)) &= D(\mathcal{T}(G)) - A(\mathcal{T}(G)) \\ &= \begin{bmatrix} 2D(G) & \\ & D(\mathcal{L}(G)) + 2I_m \end{bmatrix} - \begin{bmatrix} A(G) & R \\ R^t & A(\mathcal{L}(G)) \end{bmatrix} \\ &= \begin{bmatrix} L(G) & 0 \\ 0 & L(\mathcal{L}(G)) \end{bmatrix} + \begin{bmatrix} D(G) & -R \\ -R^t & 2I_m \end{bmatrix}. \end{aligned} \tag{17}$$

Theorem 20. *Let G be a connected graph on n vertices, m edges and smallest degree δ. Then,*

$$S_L(\mathcal{T}(G)) \geq \left| \frac{2m+2n}{n} - 2\delta \right|.$$

Proof. According to [17, Corollary 2.3], considering the Laplacian matrix $L(\mathcal{T}(G))$ partitioned as in (17), the quotient matrix becomes

$$\overline{M}_L := \overline{M}(L(\mathcal{T}(G))) = \begin{bmatrix} \frac{2m}{n} & -\frac{2m}{n} \\ -2 & 2 \end{bmatrix}.$$

The characteristic equation of $\overline{M}_L$ is

$$\left(\lambda - \frac{2m}{n}\right)(\lambda - 2) - \frac{4m}{n} = 0.$$

Solving this equation, we have

$$\lambda_1\left(\overline{M}_L\right) = \frac{2m+2n}{n} \text{ and } \lambda_2\left(\overline{M}_L\right) = 0.$$

By interlacing of the eigenvalues, [17, p. 154],

$$\mu_1(\mathcal{T}(G)) \geq \frac{2m+2n}{n} \geq \mu_{n-1}(\mathcal{T}(G)).$$

It is known that (see [13]), if H is a non-complete graph, $\mu_{n-1}(H) \leq \kappa(H)$. Moreover $\kappa(H) \leq \delta(H)$, then

$$\mu_{n-1}(H) \leq \delta(H). \tag{18}$$

Therefore, using the previous inequality with $\mathcal{T}(G)$ instead of H and recalling Remark 17 the result follows. □

5. Bounds for the spread of the total graph of a regular graph

In this section we present a lower and upper bound for the spread of the total graph of a regular graph. These bounds are obtained in function of the spread of the graph.

For an r-regular graph G, in 1973, Cvetković [9], obtained the following result.

Theorem 21. [9] Let G be a regular graph of order n and degree r with eigenvalues

$$\lambda_1 \geq \cdots \geq \lambda_n.$$

Then the eigenvalues of $\mathcal{T}(G)$ are

$$\frac{2\lambda_i + r - 2 \pm \sqrt{4\lambda_i + r^2 + 4}}{2}, i = 1, \ldots, n,$$

with multiplicity one and -2 with multiplicity $\dfrac{n(r-2)}{2}$.

Let $n \geq 2$. It is well known that if $\lambda_n(G)$ is the smallest eigenvalue of a connected graph G then

$$\lambda_n(G) \leq -1. \tag{19}$$

The following result identifies the smallest eigenvalue of the total graph of a regular graph with n vertices and degree r.

Lemma 22. *Let G be a connected regular graph of order n and degree r. Then*

$$\lambda_{\frac{n(r+2)}{2}}(\mathcal{T}(G)) = \frac{2\lambda_n + r - 2 - \sqrt{4\lambda_n + r^2 + 4}}{2}$$

where λ_n is the smallest eigenvalue of G.

Proof. Let G be a connected regular graph of order n and degree r. By the Perron-Frobenius Theory,

$$-r \leq \lambda_n.$$

From (19)

$$\lambda_n^2 + \lambda_n(r+1) + r \leq 0.$$

Now, suppose that

$$2\lambda_n + r + 2 \geq 0.$$

Also,

$$2\lambda_n + r + 2 \leq \sqrt{4\lambda_n + r^2 + 4}.$$

Therefore,

$$\frac{2\lambda_n + r - 2 - \sqrt{4\lambda_n + r^2 + 4}}{2} \leq -2. \tag{20}$$

On the other hand, if

$$2\lambda_n + r + 2 < 0,$$

then

$$\frac{2\lambda_n + r - 2 - \sqrt{4\lambda_n + r^2 + 4}}{2} < -2.$$

Since, the functions

$$f_{\pm}(x) = \frac{2x + r - 2 \pm \sqrt{4x + r^2 + 4}}{2}$$

are strictly increasing in the interval $(-r, r)$, the result follows. $\square$

Theorem 23. *Let G be a regular graph of order n and degree r. Then*

$$\frac{2S(G) + \lambda_n + 2 + \sqrt{4\lambda_n + r^2 + 4}}{2} \leq S(\mathcal{T}(G)) \leq S(G) + \sqrt{4\lambda_n + r^2 + 4} - \lambda_n.$$

Proof. If G is a regular graph then $\lambda_1(G) = r$ and $\lambda_1(\mathcal{T}(G)) = 2r$. Then

$$S(\mathcal{T}(G)) = 2r - \left(\frac{2\lambda_n + r - 2 - \sqrt{4\lambda_n + r^2 + 4}}{2}\right).$$

Since,

$$S(\mathcal{T}(G)) = \frac{2S(G) + r + 2 + \sqrt{4\lambda_n + r^2 + 4}}{2}.$$

By (20),

$$S(G) + \sqrt{4\lambda_n + r^2 + 4} - \lambda_n \geq S(\mathcal{T}(G)).$$

$\square$

Acknowledgments: The authors are grateful to the anonymous referee, whose constructive comments helped to improve the final version of this work. Enide Andrade was supported in part by the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2019. Exequiel Mallea-Zepeda was supported by Proyecto UTA-Mayor, 4740-18, Universidad de Tarapacá, Chile. M. Robbiano was partially supported by project VRIDT UCN 20180504001. E. Lenes, E. Mallea-Zepeda and J. Rodríguez also express their gratitude to professor Ricardo Reyes for his careful reading and suggestions of the final form of this work.

References

[1] E. Andrade, G. Dahl, L. Leal, M. Robbiano, New bounds for the signless Laplacian spread, Linear Algebra Appl. 566, (2019) 98-120.

[2] M. Belizad, A characterization of total graphs, Proceeding of the American Mathematical Society, 26 (1970) 383-389.

[3] M. Belizad, H. Radjavi, Structure of regular total graphs, J. London Math. Soc. 44 (1969) 433-436 MR 38 4344.

[4] A. Brouwer, W. Haemers, Spectra of graphs, Springer-Verlag, Berlin, 2012.

[5] D. M. Cardoso, D. Cvetković, P. Rowlinson, S. K. Simić, A sharp lower bound for the least eigenvalue of the signless Laplacian of a non-bipartite graph, Lin. Algebra Appl. 429 (2008) 2770-2780.

[6] D. M. Cardoso, M. Aguieiras de Freitas, E. A. Martins, M. Robbiano, Spectra of graphs obtained by a generalization of the join graph operation, Discr. Math. 313 (2013) 733-741.

[7] Y. Chen, Properties of spectra of graphs and line graphs, Appl. Math. J. Ser. B 3 (2002) 371-376.

[8] M. Cozzen, D. Moazzami, S. Stueckle, The tenacity of a graph. Proc. Seventh International Conference on the Theory and Applications of graphs, Wiley, New York, 1995, 111-1122.

[9] D. M. Cvetković, Spectrum of the total graph of a graph, Publ. Inst. Math.(Beograd), 16 49-52, 1973.

[10] D. Cvetković, P. Rowlinson, S. Simić, Eigenvalue bounds for the signless Laplacian, Publ. Inst. Math. (beograd)(N.S.) 81 (95) (2017) 11-27.

[11] D. M. Cvetković, P. Rowlinson, S. Simić, Signless Laplacians of finite graphs, Lin. Algebra Appl. 423 (2007) 155-171

[12] Y. Z. Fan, J. Xu, Y. Wang, D. Liang, The Laplacian spread of a tree, Discr. Math. Theor. Comput. Sci. 10 (2008) 79-86.

[13] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (1973) 298-305.

[14] D. A. Gregory, D. Heshkowitz, S. J. Kirkland, The spread of the spectrum of a graph, Linear Algebra Appl., 332-334 (2001) 23-35.

[15] R. Grone, R. Merris, V. S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1990) 218-238.

[16] I. Gutman, K. C. Das, The first Zagreb index 30 years after, MATCH Comm. Math. Comput. Chem. 50 (2004) 82-92.

[17] W. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 227-228 (1995) 593-616.

[18] F. Harary, Graph theory, Reading, MA Addison-Wesley, p. 43, 1994.

[19] S. Hwang, Cauchy's interlace theorem for eigenvalues of hermitian matrices, The American Mathematical Monthly, Vol. 111, 157-159, (2004).

[20] E. Lenes, Exequiel Mallea-Zepeda, María Robbiano, Jonnathan Rodríguez Z., On line graphs with maximum energy, Linear Algebra and its Appl., 545 (2018) 15-31.

[21] M. Liu, B. Liu, The signless Laplacian spread, Linear Algebra and Appl. 432 (2010) 505-514.

[22] C. S. Oliveira, L. S. de Lima, N. M. M. Abreu, S. Kirkland, Bounds on the Q-spread of a graph, Linear Algebra Appl. 432, 9, (2010) 2342-2351.

[23] H. Whitney, Congruent graphs and the connectivity of graphs, Amer. J. Math. 54, (1932) 150-168.

[24] L. Yinkut, W. Zongtian, Y. Xiaokut, L. Erquiang, Tenacity of total graphs, Int. J. Found. Comput. Sci. 25, (2014) 553.